

## **THE LAW OF A STOCHASTIC INTEGRAL WITH RESPECT TO SUBFRACTIONAL BROWNIAN MOTIONS**

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### **Abstract**

In this note, we consider the distribution of the random variable  $\int_0^T S_s^\alpha dS_s^H$  and obtain the expression of its characteristic function, where  $S^\alpha$  and  $S^H$  are two independent sub-fractional Brownian motions with indices  $\alpha \in (0, 1)$  and  $H \in (\frac{1}{2}, 1)$ , respectively.

### **1. Introduction**

Recently, as a generalization of Brownian motion, Bojdecki et al. [5] introduced and studied a rather special class of self-similar Gaussian  
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processes, which preserve many properties of the fractional Brownian motion (fBm in short). This process arises from occupation time fluctuations of branching particle systems with Poisson initial condition, which is called the *sub-fractional Brownian motion*. The so-called sub-fractional Brownian motion (sub-fBm in short) with index  $H \in (0, 1)$  is a mean zero Gaussian process  $S^H = \{S_t^H, t \geq 0\}$  with  $S_0^H = 0$ , and the covariance

$$R_H(t, s) \equiv E[S_t^H S_s^H] = s^{2H} + t^{2H} - \frac{1}{2}[(s+t)^{2H} + |t-s|^{2H}], \quad (1.1)$$

for all  $s, t \geq 0$ . For  $H = 1/2$ ,  $S^H$  coincides with the standard Brownian motion  $B$ .  $S^H$  is neither a semimartingale nor a Markov process unless  $H = 1/2$ , so many of the powerful techniques from stochastic analysis are not available, when dealing with  $S^H$ . The sub-fBm has properties analogous to those of fBm (self-similarity, long-range dependence, Hölder paths), and satisfies the following estimates:

$$[(2 - 2^{2H-1}) \wedge 1](t-s)^{2H} \leq E[(S_t^H - S_s^H)^2] \leq [(2 - 2^{2H-1}) \vee 1](t-s)^{2H}. \quad (1.2)$$

But, its increments are not stationary, more works for sub-fBm can be found in Bojdecki et al. [6], Tudor [15, 16, 17, 18, 19], and Yan-Shen [21]. In this note, we consider the law of stochastic integral

$$\int_0^T S_t^\alpha dS_t^H, \quad (1.3)$$

where  $S_t^\alpha$  and  $S_t^H$  are two independent sub-fBms. Our aim is to obtain the expression of its characteristic function.

We have known that it is difficult to compute the law of a stochastic integral with respect to the Wiener process, when the integrand is not deterministic. The systematic study for this problem was initiated in Lévy [10]. He showed that the characteristic function of  $A_t = \int_0^t X_s dY_s - \int_0^t Y_s dX_s$  is

$$E(e^{iuA_t}) = \frac{1}{\cosh(ut)}, \quad t \geq 0, u \in \mathbb{R},$$

where  $(X_t, Y_t)$  is an  $\mathbb{R}^2$ -valued Brownian motion with  $(X_0, Y_0) = (0, 0)$ . This is called *Lévy's stochastic area formula*. Berthuet [3] and Yor [22] (see also Protter [14]) gave other proof, and considered the law of the random variables

$$\lambda \int_0^t X_s dY_s + \rho \int_0^t Y_s dX_s, \quad t \geq 0.$$

The two-parameter case was considered in Julià-Nualart [9] and Nualart [12]. Yan-Chen [20] considered the intersection local time and calculus for the stochastic area process  $A_t$ . The stochastic area process  $A_t$  shares some properties of Brownian motion. For example,  $A_t$  satisfies a *reflection principle*. If one changes the sign of the increments of  $A_t$  after a stopping time, the process obtained thereby has the same distribution as that of  $A_t$ . One can use this fact to show, for example, that if  $S_t = \sup_{0 \leq s \leq t} A_s$ , then  $S_t$  has the same distribution as  $|A_t|$ , for  $t > 0$ .

As an extension, recently, Bardina-Tudor [2] considered a similar integral driven by fractional Brownian motions, and they obtained the characteristic function of the random variable  $S = \int_0^1 B_s^\alpha dB_s^H$ , where  $B^\alpha$  and  $B^H$  are two independent fractional Brownian motions with Hurst indexes  $\alpha \in (0, 1)$  and  $H \in (\frac{1}{2}, 1)$ , respectively. As is well-known, in recent years, the long-range dependence property has become an important aspect of stochastic models in various scientific areas including hydrology, telecommunication, turbulence, image processing, and finance. The best known and most widely used process that exhibits the long-range dependence property is *fractional Brownian motion*. The fBm is a suitable generalization of the standard Brownian motion, but exhibits long-range dependence, self-similarity, and stationary increments. It is impossible to list here all the contributors in previous topics. Some

surveys and complete literatures could be found in Biagini et al. [4], Hu [8], Mishura [11], Nualart [13]. However, contrast to the extensive studies on fBm, there has been little systematic investigation on other self-similar Gaussian processes. The main reasons are the complexity of dependence structures and the non-availability of convenient stochastic integral representations for self-similar Gaussian processes, which do not have stationary increments. On the other hand, many authors have proposed to use more general self-similar Gaussian processes and random fields as stochastic models, and such applications have raised many interesting theoretical questions about self-similar Gaussian processes and fields in general. Thus, it seems interesting to study the law of stochastic integrals driven by more general self-similar Gaussian processes.

This note is organized as follows. In Section 2, we present some preliminaries for sub-fBm and the Wiener integral with respect to sub-fBm. In Section 3, we obtained the characteristic function of stochastic integral  $\int_0^T S_t^\alpha dS_t^H$ . The case of two-parameter is considered in Section 4.

## 2. Preliminaries on Sub-FBM

Let  $\{S_t^H, t \in [0, T]\}$  be a sub-fBm with  $\frac{1}{2} < H < 1$ , defined on the complete probability space  $(\Omega, \mathcal{F}, P)$ . It is possible to construct a stochastic calculus of variations with respect to the Gaussian process  $S^H$ , which will be related to the Malliavin calculus. Some surveys and complete literatures could be found in Alòs et al. [1], Nualart [13], and Tudor [18]. We recall here the basic definitions and results of this calculus. The crucial ingredient is the canonical Hilbert space  $\mathcal{H}_H$  (is also said to be reproducing kernel Hilbert space) associated to the sub-fBm  $S^H$ , which is defined as the closure of the linear space  $\mathcal{E}$  generated by the indicator functions  $\{1_{[0,t]}, t \in [0, T]\}$  with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}_H} = \int_0^t \int_0^s \phi_H(u, v) du dv = R_H(t, s),$$

where  $\phi_H(s, t) = H(2H - 1)[|s - t|^{2H-2} - (s + t)^{2H-2}]$ . As usual, we can define the linear application

$$\Phi : \mathcal{E} \rightarrow L^2(\Omega, \mathcal{F}, P),$$

by

$$1_{[0,t]} \mapsto \Phi(1_{[0,t]}) = \int_0^T 1_{[0,t]}(s) dS_s^H \equiv S_t^H. \quad (2.1)$$

The application can be extended to a linear isometry between  $\mathcal{H}_H$  and the Gaussian space associated with  $S^H$ . We will denote the isometry by  $\varphi \rightarrow S^H(\varphi)$ , and

$$\langle \varphi, \psi \rangle_{\mathcal{H}_H} = E[S^H(\varphi)S^H(\psi)] = \int_0^T \int_0^T \varphi(s)\psi(t)\phi_H(s, t) ds dt,$$

for any  $\varphi, \psi \in \mathcal{H}_H$ . We call  $S^H(\varphi) := \int_0^T \varphi(s) dS_s^H$ , the Wiener integral of  $\varphi$  with respect to  $S^H$ . Sometimes working with the space  $\mathcal{H}_H$  is not convenient; once, because this space may contain also distributions (see, for example, Tudor [16, 19] for more details) and twice, because the norm in this space is not always tractable. We will use the subspace  $|\mathcal{H}_H|$  of  $\mathcal{H}_H$ , which is defined as the set of measurable functions  $\varphi : [0, T] \mapsto \mathbb{R}$  on  $[0, T]$  such that

$$\int_0^T \int_0^T |\varphi(s)||\varphi(t)|\phi_H(s, t) du dv < \infty.$$

It has been proved in Tudor [19] that  $|\mathcal{H}|_H$  is a strict subspace of  $\mathcal{H}_H$ .

For  $\frac{1}{2} < H < 1$ , we denote by  $\mathcal{S}$  the set of smooth functionals of the form

$$F = f(S^H(\varphi_1), \dots, S^H(\varphi_n)),$$

where  $f \in C_b^\infty(\mathbb{R}^n)$  and  $\varphi_i \in \mathcal{H}_H$ . The Malliavin derivative of a functional  $F$  defined as above is given by

$$D^H F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(S^H(\varphi_1), \dots, S^H(\varphi_n)) \varphi_i.$$

The derivative operator  $D^H$  is then a closable operator from  $L^2(\Omega)$  into  $L^2(\Omega, \mathcal{H}_H)$ . We denote by  $\mathbb{D}^{1,2}$  the closure of  $\mathcal{S}$  with respect to the norm

$$\|F\|_{1,2} := \sqrt{E|F|^2 + E\|DF\|_{\mathcal{H}}^2}.$$

The divergence integral  $\delta^H$  is the adjoint operator of  $D^H$ . That is, we say that a random variable  $u$  in  $L^2(\Omega, \mathcal{H}_H)$  belongs to the domain of the divergence operator  $\delta^H$ , denoted by  $\text{Dom}(\delta^H)$ , if

$$E|\langle D^H F, u \rangle_{\mathcal{H}_H}| \leq c\|F\|_{L^2(\Omega)},$$

for every  $F \in \mathcal{S}$ . In this case,  $\delta^H(u)$  is defined by the duality relationship

$$E[F\delta^H(u)] = E\langle D^H F, u \rangle_{\mathcal{H}_H}, \quad (2.2)$$

for any  $u \in \mathbb{D}^{1,2}$ , and we have the following integration by parts formula:

$$F\delta^H(u) = \delta^H(Fu) + \langle D^H F, u \rangle_{\mathcal{H}_H}, \quad (2.3)$$

for any  $u \in \text{Dom}(\delta^H)$ ,  $F \in \mathbb{D}^{1,2}$  such that  $Fu \in L^2(\Omega, \mathcal{H}_H)$ . It follows that

$$E[\delta^H(u)^2] = E\|u\|_{\mathcal{H}_H}^2 + E\langle D^H u, (D^H u)^* \rangle_{\mathcal{H}_H \otimes \mathcal{H}_H},$$

where  $(D^H u)^*$  is the adjoint of  $D^H u$  in the Hilbert space  $\mathcal{H}_H \otimes \mathcal{H}_H$ , and

$$\|u\|_{\mathcal{H}_H}^2 = \int_0^T \int_0^T u_s u_r \phi_H(s, r) ds dr, \tag{2.4}$$

and for  $\varphi : [0, T]^2 \rightarrow \mathbb{R}$ , we have

$$\|\varphi\|_{\mathcal{H}_H \otimes \mathcal{H}_H}^2 = \int_{[0, T]^4} \varphi(t, s) \varphi(t', s') \phi_H(t, t') \phi_H(s, s') dt ds dt' ds'.$$

We also will use the notation

$$\delta^H(u) = \int_0^T u_s dS_s^H,$$

to express the Skorohod integral of a process  $u$ . It is also possible to introduce multiple integrals  $I_n(f_n)$ ,  $f_n \in \mathcal{H}_H^{\otimes n}$  with respect to  $S^H$ . For the divergence integral, we have the following convergence: if  $\{u_n\}$  is a sequence of elements in  $\text{Dom}(\delta^H)$  such that  $u_n \rightarrow u$  in  $L^2(\Omega, \mathcal{H}_H)$ , and  $\delta^H(u_n) \rightarrow G$  in  $L^2(\Omega)$ , then we have  $u \in \text{Dom}(\delta^H)$  and  $\delta^H(u) = G$ .

Clearly, for any  $f \in \mathcal{H}_H$ , the Wiener integral with respect to  $S^H$  can be defined as (see Tudor [19])

$$\int_0^T f(s) dS_s^H = \lim_{n \rightarrow \infty} \sum_{j=1}^n f(s_j) (S_{s_j}^H - S_{s_{j-1}}^H), \tag{2.5}$$

where  $\{0 = s_0, s_1, \dots, s_n = T\}$  is a partition of  $[0, T]$  such that  $\max_i \{s_{i+1} - s_i\} \rightarrow 0$ , as  $n$  tends to infinity. Moreover, if stochastic process  $u$  is independent of  $S^H$ , then the Skorohod integral  $\int_0^T u(s) dS_s^H$  can be defined as (2.5), since the Malliavin derivative of  $u$  is zero.

Finally, we will denote by  $|\mathcal{H}|_{H,S}^n$  the set of symmetric functions in  $|\mathcal{H}|_H^n$ . For  $f \in |\mathcal{H}|_{H,S}^2$ , we recall the Hilbert-Schmidt operator (see Caithamer [7])  $K_f^H : |\mathcal{H}|_H^2 \rightarrow |\mathcal{H}|_H^2$  given by

$$(K_f^H \psi)(y) = \int_0^T \int_0^T f(x, y) \psi(x') \phi_H(x, x') dx dx'. \quad (2.6)$$

One can easily check that, if  $f$  is positive and  $H \in (\frac{1}{2}, 1)$ , then the eigenvalues of operator  $K_f^H$  are positive. In fact, we can write

$$(K_f^H \psi)(y) = \int_0^T A(x', y) \psi(x') dx',$$

where  $A(x', y) = \int_0^T f(x, y) \phi_H(x, x') dx$  is positive, then the operator  $K_f^H$  is a positive operator. It is noteworthy that the operator  $K_f^H$  will be changed as

$$K_f^H \varphi(y) = \int_0^T f(x, y) \varphi(x) dx,$$

provided  $H = \frac{1}{2}$ .

### 3. The Characteristic Function of the Integral (1.3)

Throughout this section,  $S^H$  and  $S^\alpha$  will denote two independent sub-fractional Brownian motions with parameters  $H$  and  $\alpha$ , respectively. We compute the characteristic function of the random variable

$$\ell = \int_0^T S_t^\alpha dS_t^H. \quad (3.1)$$

The method used here is essentially due to Bardina-Tudor [2]. Note that, when  $H \in (\frac{1}{2}, 1)$ , the random variable  $\ell$  defined in (3.1) is well-defined,



since obviously  $S^\alpha$  belongs to  $L^2(\Omega) \times |\mathcal{H}|_H$  for any  $\alpha$ . The main object of this section is to explain and prove the following theorem.

**Theorem 3.1.** *Let  $\alpha \in (\frac{1}{2}, 1)$ ,  $H \in (\frac{1}{2}, 1)$ . If the random variable  $\ell$  is given by (3.1), then we have*

$$Ee^{it\ell} = \prod_{j \geq 1} \left( 1 + \frac{1}{t^2 \mu_j} \right)^{\frac{1}{2}}, \quad (3.2)$$

where  $\mu_j$ ,  $j \geq 1$  are the eigenvalues of the operator  $K_{f^H}^\alpha$  given by (2.6) and  $f^H$  is defined by (3.4).

In order to prove the theorem, we need some lemmas.

**Lemma 3.1.** *Assume  $H \in (\frac{1}{2}, 1)$  and  $\alpha \in (0, 1)$ . Denote by*

$$\ell_n = \sum_{k=0}^{n-1} S_{t_k}^\alpha (S_{t_{k+1}}^H - S_{t_k}^H), \quad (3.3)$$

where  $\Delta_n := \{0 = t_0 < t_1 < t_2 < \dots < t_n = T\}$  denotes a partition of  $[0, T]$ , such that  $|\Delta|_n = \max_{n \geq 1} |t_n - t_{n-1}|$  tends to zero as  $n$  tends to infinity. We then have

$$\ell_n \rightarrow \ell,$$

in  $L^2(\Omega)$  as  $n$  tends to  $\infty$ .

**Proof.** By the independence of  $S^H$  and  $S^\alpha$ , we can write

$$S_{t_k}^\alpha (S_{t_{k+1}}^H - S_{t_k}^H) = \int_{t_k}^{t_{k+1}} S_{t_k}^\alpha dS_r^H.$$

Now, it suffices to prove

$$\sum_{k=0}^{n-1} S_{t_k}^\alpha 1_{[t_k, t_{k+1}]}(\cdot) \rightarrow S^\alpha,$$

in  $L^2(\Omega) \times |\mathcal{H}|_H$  as  $n$  tends to infinity. Actually, in general, to prove the convergence of a sequence of stochastic integrals of divergence type, one needs the convergence of the Malliavin derivatives, but in our case, it is unnecessary due to the independence of the two sub-fBms. Noting that  $S^\alpha = \sum_{k=0}^{n-1} S^\alpha 1_{[t_k, t_{k+1}]}(\cdot)$  for all  $n \geq 1$ , by (1.2), we have

$$\begin{aligned}
& E \left\| \sum_{k=0}^{n-1} (S_{t_k}^\alpha - S^\alpha) 1_{[t_k, t_{k+1}]} \right\|_{|\mathcal{H}|_H}^2 \\
&= \sum_{k,j=0}^{n-1} \int_{t_k}^{t_{k+1}} \int_{t_j}^{t_{j+1}} E(S_{t_k}^\alpha - S_u^\alpha) (S_{t_j}^\alpha - S_v^\alpha) \phi_H(u, v) dudv \\
&\leq [(2 - 2^{2\alpha-1}) \vee 1] \sum_{k,j=0}^{n-1} \int_{t_k}^{t_{k+1}} \int_{t_j}^{t_{j+1}} |t_k - u|^\alpha |t_j - v|^\alpha \phi_H(u, v) dudv \\
&= [(2 - 2^{2\alpha-1}) \vee 1] (2 - 2^{2H-1}) T^{2H} |\Delta|_n^{2\alpha} \rightarrow 0,
\end{aligned}$$

as  $n$  tends to infinity. □

**Lemma 3.2.** (1) Let  $\alpha \in (\frac{1}{2}, 1)$ . Consider the function

$$\begin{aligned}
f^H(x, y) &= -2^{2H-1} T^{2H} + \frac{1}{2} \{ (T+x)^{2H} + (T-x)^{2H} + (T+y)^{2H} \\
&\quad + (T-y)^{2H} - (x+y)^{2H} - |x-y|^{2H} \}, \quad x, y \in [0, T], \quad (3.4)
\end{aligned}$$

with  $\frac{1}{2} < H < 1$ . Then, we have  $f^H \in |\mathcal{H}|_{\alpha, S}^2$ .

(2) Let  $H \in (\frac{1}{2}, 1)$ . Consider the function

$$f^\alpha(x, y) = x^{2\alpha} + y^{2\alpha} - \frac{1}{2}[(x+y)^{2\alpha} + |x-y|^{2\alpha}], \quad x, y \in [0, T], \quad (3.5)$$

with  $0 < \alpha < 1$ . Then, we have  $f^\alpha \in |\mathcal{H}|_{H,S}^2$ .

**Proof.** Let us prove the point (1) and the point (2) is similar. We have to show that

$$I := \int_{[0,T]^4} f^H(x_1, y_1) f^H(x_2, y_2) \phi_\alpha(x_1, x_2) \phi_\alpha(y_1, y_2) dx_1 dx_2 dy_1 dy_2 < \infty.$$

Noting that

$$\begin{aligned} f^H(x_i, y_i) &= E(S_T^H - S_{x_i}^H) (S_T^H - S_{y_i}^H) \\ &\leq \left( E(S_T^H - S_{x_i}^H)^2 \right)^{\frac{1}{2}} \left( E(S_T^H - S_{y_i}^H)^2 \right)^{\frac{1}{2}} \\ &\leq (T - x_i)^H (T - y_i)^H, \end{aligned}$$

by (1.2), we see that the integral  $I$  is bounded by

$$I \leq \int_{[0,T]^4} \prod_{k=1}^2 [(T - x_k)^H (T - y_k)^H] \phi_\alpha(x_1, x_2) \phi_\alpha(y_1, y_2) dx_1 dx_2 dy_1 dy_2,$$

where

$$\begin{aligned} \phi_\alpha(x_1, x_2) &= \alpha(2\alpha - 1) \left[ |x_1 - x_2|^{2\alpha-2} - (x_1 + x_2)^{2\alpha-2} \right] \\ &\leq \alpha(2\alpha - 1) |x_1 - x_2|^{2\alpha-2}, \end{aligned}$$

for all  $x_1, x_2 \geq 0$ . It follows that

$$\begin{aligned} I &\leq (\alpha(2\alpha - 1))^2 \int_{[0,T]^4} (T - x_1)^H (T - y_1)^H (T - x_2)^H (T - y_2)^H \\ &\quad \cdot |x_1 - x_2|^{2\alpha-2} |y_1 - y_2|^{2\alpha-2} dx_1 dx_2 dy_1 dy_2 \\ &= \left[ \alpha(2\alpha - 1) \int_{[0,T]^2} (T - x_1)^H (T - x_2)^H |x_1 - x_2|^{2\alpha-2} dx_1 dx_2 \right]^2. \end{aligned}$$

Make substitution  $z = \frac{x-y}{T-y}$ . Then, we have

$$\begin{aligned}
& \int_{[0, T]^2} (T-x)^H (T-y)^H |x-y|^{2\alpha-2} dx dy \\
&= 2 \int_0^T \int_0^x (T-x)^H (T-y)^H |x-y|^{2\alpha-2} dx dy \\
&= 2 \int_0^T \int_0^{\frac{x}{T}} (T-x)^{2H+2\alpha-1} (1-z)^{-H-2\alpha} z^{2\alpha-2} dz dx \\
&= 2 \int_0^1 (1-z)^{-H-2\alpha} z^{2\alpha-2} \left( \int_{Tz}^T (T-x)^{2H+2\alpha-1} dx \right) dz \\
&= \frac{T^{2H+2\alpha}}{H+\alpha} \int_0^1 (1-z)^H z^{2\alpha-2} dz \\
&= \frac{T^{2H+2\alpha}}{H+\alpha} \mathbf{B}(2\alpha-1, H+1) < \infty,
\end{aligned}$$

because of  $\alpha \in (\frac{1}{2}, 1)$ , where  $\mathbf{B}$  is the Beta function. This completes the proof.  $\square$

Finally, we recall the following fact.

**Lemma 3.3.** *If  $X, Y$  are two independent random vectors, then*

$$E[\Psi(X, Y)|X] = \rho(X),$$

where  $\rho(x) = E[\Psi(x, Y)]$  and  $\Psi$  is a measurable function.

Now, we can prove our main result.

**Proof of Theorem 3.1.** By Lemma 3.1, we have

$$Ee^{it\ell} = \lim_{n \rightarrow \infty} Ee^{it\ell_n},$$

where  $\ell_n$  is given by (3.3) with  $t_k = \frac{k}{n}T$ , for every  $k = 0, 1, \dots, n-1$ .

Let us compute the characteristic function of the random variable  $\ell_n$ .

Define  $X = (X_0, X_1, \dots, X_{n-1})$  and  $Y = (Y_0, Y_1, \dots, Y_{n-1})$  by

$$X_k = S_{\frac{k}{n}T}^\alpha \quad \text{and} \quad Y_k = S_{\frac{k+1}{n}T}^H - S_{\frac{k}{n}T}^H, \quad k = 0, 1, 2, \dots, n-1. \quad (3.6)$$

Thanks to Lemma 3.3 with  $\psi(x, y) = \exp\{it\sum_{j=1}^n x_j y_j\}$ , we obtain

$$\rho(x) = E\left(e^{it\sum_{k=0}^{n-1} x_k Y_k}\right) = e^{-\frac{1}{2}t^2 x^T A^H x},$$

and

$$E(e^{it\ell_n}) = E\rho(X) = E\left(e^{-\frac{1}{2}t^2 X^T A^H X}\right) \equiv E\left(e^{-\frac{1}{2}t^2 R_n}\right),$$

where  $R_n = X^T A^H X$ , and the matrix  $A^H = (A_{k,l}^H)_{k,l=0,1,\dots,n-1}$  is given by

$$\begin{aligned} A_{k,l}^H &= E\left(S_{\frac{k+1}{n}T}^H - S_{\frac{k}{n}T}^H\right)\left(S_{\frac{l+1}{n}T}^H - S_{\frac{l}{n}T}^H\right) \\ &= \frac{T^{2H}}{2n^{2H}}[-(k+l+2)^{2H} - 2|k-l|^{2H} + 2(k+l+1)^{2H} \\ &\quad - (k+l)^{2H} + |k+1-l|^{2H} + |k-l-1|^{2H}]. \end{aligned}$$

A straightforward calculation shows that

$$\begin{aligned} R_n &= \sum_{k,l=0}^{n-1} A_{k,l}^H S_{\frac{k}{n}T}^\alpha S_{\frac{l}{n}T}^\alpha \\ &= \sum_{k,l=1}^{n-1} A_{k,l}^H \left(\sum_{k'=0}^{k-1} \left(S_{\frac{k'+1}{n}T}^\alpha - S_{\frac{k'}{n}T}^\alpha\right)\right) \left(\sum_{l'=0}^{l-1} \left(S_{\frac{l'+1}{n}T}^\alpha - S_{\frac{l'}{n}T}^\alpha\right)\right) \\ &= \sum_{k',l'=1}^{n-2} \left(S_{\frac{k'+1}{n}T}^\alpha - S_{\frac{k'}{n}T}^\alpha\right) \left(S_{\frac{l'+1}{n}T}^\alpha - S_{\frac{l'}{n}T}^\alpha\right) \sum_{l=l'+1}^{n-1} \sum_{k=k'+1}^{n-1} A_{k,l}^H. \end{aligned}$$

We now claim that reduces the expression  $R_n$ . We have

$$\begin{aligned}
& \sum_{l=l'+1}^{n-1} \sum_{k=k'+1}^{n-1} A_{k,l}^H \\
&= \frac{T^{2H}}{2n^{2H}} \sum_{l=l'+1}^{n-1} \sum_{k=k'+1}^{n-1} [- (k+l+2)^{2H} + 2(k+l+1)^{2H} - (k+l)^{2H} \\
&\quad - 2|k-l|^{2H} + |k+1-l|^{2H} + |k-l-1|^{2H}] \\
&= \frac{T^{2H}}{2n^{2H}} \sum_{l=l'+1}^{n-1} [-(n+l+1)^{2H} + (k'+l+2)^{2H} + (n+l)^{2H} - (k'+l+1)^{2H} \\
&\quad + (n-l)^{2H} - |k'+1-l|^{2H} - |n-l-1|^{2H} + |k'-l|^{2H}] \\
&= \frac{T^{2H}}{2n^{2H}} [-(2n)^{2H} + (n+l'+1)^{2H} + (n-l'-1)^{2H} + (n+k'+1)^{2H} \\
&\quad - (k'+l'+2)^{2H} - |k'-l'|^{2H} - |n-k'-1|^{2H}] \\
&= -2^{2H-1} T^{2H} + \frac{1}{2} \left[ \left( \frac{n+l'+1}{n} T \right)^{2H} + \left( \frac{n-l'-1}{n} T \right)^{2H} + \left( \frac{n+k'+1}{n} T \right)^{2H} \right. \\
&\quad \left. + \left( \frac{n-k'-1}{n} T \right)^{2H} - \left( \frac{k'+1+l'+1}{n} T \right)^{2H} - \left( \frac{k'-l'}{n} T \right)^{2H} \right] \\
&= f^H \left( \frac{k'+1}{n} T, \frac{l'+1}{n} T \right),
\end{aligned}$$

where the function  $f^H$  is given by (3.4). Combining the above calculations lead to

$$R_n = \sum_{k,l=0}^{n-1} f^H \left( \frac{k+1}{n} T, \frac{l+1}{n} T \right) \left( S_{\frac{k+1}{n}T}^\alpha - S_{\frac{k}{n}T}^\alpha \right) \left( S_{\frac{l+1}{n}T}^\alpha - S_{\frac{l}{n}T}^\alpha \right).$$

Denote by  $\mu_j, j \geq 1$ , the eigenvalues of the operator  $K_{f^H}^\alpha$  and by  $g_j, j \geq 1$ , the corresponding eigenfunctions. Then,  $g_j, j \geq 1$  are

orthonormal in  $|\mathcal{H}|_{H,S}$  and the  $\mu_j, j \geq 1$  are square-summable, which implies that random variables

$$B_j := \int_0^T g_j(s) dS_s^\alpha, \quad j \geq 1,$$

are independent standard normal random variables, and moreover, we can write

$$f^H(x, y) = \sum_{j \geq 1} \mu_j g_j(x) g_j(y), \quad (3.7)$$

for  $x, y \in [0, T]$ , by using the Lemma 3.2. Thus, the sum  $R_n$  becomes

$$\begin{aligned} R_n &= \sum_{k,l=0}^{n-1} \left( \sum_{j \geq 1} \mu_j g_j \left( \frac{k+1}{n} T \right) g_j \left( \frac{l+1}{n} T \right) \right) \left( S_{\frac{k+1}{n}T}^\alpha - S_{\frac{k}{n}T}^\alpha \right) \left( S_{\frac{l+1}{n}T}^\alpha - S_{\frac{l}{n}T}^\alpha \right) \\ &= \sum_{j \geq 1} \mu_j \left( \sum_{k=0}^{n-1} g_j \left( \frac{k+1}{n} T \right) \left( S_{\frac{k+1}{n}T}^\alpha - S_{\frac{k}{n}T}^\alpha \right) \right)^2. \end{aligned}$$

Combining this with

$$\sum_{k=0}^{n-1} g_j \left( \frac{k+1}{n} T \right) \left( S_{\frac{k+1}{n}T}^\alpha - S_{\frac{k}{n}T}^\alpha \right) \rightarrow \int_0^T g_j(s) dS_s^\alpha \equiv B_j,$$

in  $L^2(\Omega)$  for all  $j \geq 1$ , as  $n$  tends to infinity, we get

$$R_n \rightarrow \sum_{j \geq 1} \mu_j (B_j)^2,$$

in  $L^1(\Omega)$ , as  $n$  tends to infinity, which deduces

$$\begin{aligned} E(e^{it\ell}) &= \lim_{n \rightarrow \infty} E e^{-\frac{t^2}{2} R_n} = E \left( e^{-\frac{t^2}{2} \sum_{j \geq 1} \mu_j B_j^2} \right) \\ &= \prod_{j \geq 1} E \left( e^{-\frac{t^2}{2} \mu_j B_j^2} \right) = \prod_{j \geq 1} \left( \frac{1}{1 + t^2 \mu_j} \right)^{\frac{1}{2}}, \end{aligned}$$

since the eigenvalues are positive. This completes the proof.  $\square$

We can state an alternative result that allows to consider the situation, when the parameter of the integrand  $\alpha$  is less than  $\frac{1}{2}$ .

**Theorem 3.2.** *Let  $H \in (\frac{1}{2}, 1)$  and  $\alpha \in (0, \frac{1}{2})$ . Then, the characteristic function of the random variable  $\ell$  given by (3.1) is*

$$Ee^{it\ell} = \prod_{j \geq 1} \left( 1 + \frac{1}{t^2 \nu_j} \right)^{\frac{1}{2}}, \quad (3.8)$$

where  $\nu_j, j \geq 1$  are the eigenvalues of the operator  $K_{f^\alpha}^H$  given by (2.6) and  $f^\alpha$  is defined by (3.5).

**Proof.** We follow the lines of this corollary by interchanging the roles of  $X$  and  $Y$  in (3.6). We obtain that

$$E(e^{it\ell}) = \lim_{n \rightarrow \infty} E \left( e^{-\frac{t^2}{2} R_n} \right),$$

where

$$\begin{aligned} R_n &:= \sum_{k,l=0}^{n-1} E \left( S_{\frac{k}{n}T}^\alpha S_{\frac{l}{n}T}^\alpha \right) \left( S_{\frac{k+1}{n}T}^H - S_{\frac{k}{n}T}^H \right) \left( S_{\frac{j+1}{n}T}^H - S_{\frac{j}{n}T}^H \right) \\ &= \sum_{k,l=0}^{n-1} f^\alpha \left( \frac{k}{n}T, \frac{l}{n}T \right) \left( S_{\frac{k+1}{n}T}^H - S_{\frac{k}{n}T}^H \right) \left( S_{\frac{j+1}{n}T}^H - S_{\frac{j}{n}T}^H \right), \end{aligned}$$

and  $f^\alpha$  is given by (3.5). The rest of the proof is same as this of Theorem 3.1.  $\square$

#### 4. The Case of Two-parameter

In this section, we will discuss the case of sub-fractional Brownian sheet. A sub-fractional Brownian sheet  $S^{\alpha_1, \alpha_2}(s, t)$  with parameters  $\alpha_1, \alpha_2 \in (0, 1)$  is a centered Gaussian process starting from zero with the covariance function given by



$$\begin{aligned}
R_{\alpha_1, \alpha_2}((s, t), (u, v)) &= E(S^{\alpha_1, \alpha_2}(s, t)S^{\alpha_1, \alpha_2}(u, v)) \\
&= \left( s^{2\alpha_1} + u^{2\alpha_1} - \frac{1}{2}[(s+u)^{2\alpha_1} + |s-u|^{2\alpha_1}] \right) \\
&\quad \cdot \left( t^{2\alpha_2} + v^{2\alpha_2} - \frac{1}{2}[(t+v)^{2\alpha_2} + |t-v|^{2\alpha_2}] \right). \quad (4.1)
\end{aligned}$$

Denote by  $S^{\alpha_1, \alpha_2}(s, t)$  and  $S^{H_1, H_2}(s, t)$  two independent sub-fractional Brownian sheets. We denote  $\mathcal{H}(\alpha_1, \alpha_2)$ , the canonical Hilbert space of the Gaussian process  $S^{\alpha_1, \alpha_2}(s, t)$  defined by the closure of the linear vector space generated by the elementary functions  $\{1_{[0, t] \times [0, s]}, s, t \in [0, T]\}$  with respect to the inner product

$$\langle 1_{[0, t] \times [0, s]}, 1_{[0, u] \times [0, v]} \rangle_{\mathcal{H}(\alpha_1, \alpha_2)} = R_{\alpha_1, \alpha_2}((s, t), (u, v)). \quad (4.2)$$

If one of  $\alpha_1$  and  $\alpha_2$  is greater than  $\frac{1}{2}$ , then the space  $\mathcal{H}(\alpha_1, \alpha_2)$  may contain distributions. In this case, it is more convenient to work with the following set of functions  $|\mathcal{H}|(\alpha_1, \alpha_2)$ , which is given as  $|\mathcal{H}|(\alpha_1, \alpha_2) = |\mathcal{H}|_{\alpha_1} \otimes |\mathcal{H}|_{\alpha_2}$ . Therefore, Wiener integrals with respect to  $S^{\alpha_1, \alpha_2}(s, t)$  can be naturally defined for integrands in  $|\mathcal{H}|(\alpha_1, \alpha_2)$ . The following theorem is our main result in this section.

**Theorem 4.1.** *Let  $H_1, H_2, \alpha_1, \alpha_2 > \frac{1}{2}$ . Then, the characteristic function of the random variable*

$$U = \int_0^T \int_0^T S^{\alpha_1, \alpha_2}(s, t) dS^{H_1, H_2}(s, t), \quad (4.3)$$

is given by

$$E(e^{itU}) = \prod_{j, k \geq 1} \sqrt{\frac{1}{1 + t^2 \mu_{j, 1} \mu_{k, 2}}}, \quad (4.4)$$

where  $\mu_{k, r}, k \geq 1$  are the eigenvalues of the operators  $K_{f^{H_r}}^{\alpha_r}$  with  $f^{H_r}$  given by (3.4), for  $r = 1, 2$ .

**Proof.** The proof of the theorem is same as that of Theorem 3.1. Denote by

$$A_n = \sum_{k,l=0}^{n-1} S^{\alpha_1, \alpha_2}(t_k, t_l) S^{H_1, H_2}(\Delta_{k,l}), \quad (4.5)$$

where

$$\begin{aligned} S^{H_1, H_2}(\Delta_{k,l}) &= S^{H_1, H_2}(t_{k+1}, t_{l+1}) - S^{H_1, H_2}(t_k, t_{l+1}) \\ &\quad - S^{H_1, H_2}(t_{k+1}, t_l) + S^{H_1, H_2}(t_k, t_l), \end{aligned}$$

with  $t_k = \frac{k}{n}T$  for every  $k = 0, 1, \dots, n-1$ . As in the proof of Lemma 3.1, we can prove that  $A_n \rightarrow U$  as  $n$  tends to infinity in  $L^2(\Omega)$  for  $H_r > \frac{1}{2}$ , and  $\alpha_r > \frac{1}{2}$ ,  $r = 1, 2$ . Following the same line of reasoning as in the proof of Lemma 3.1, we have

$$E(e^{itA_n}) = E\left(e^{-\frac{1}{2}t^2 B_n}\right) \quad \text{and} \quad E(e^{itU}) = \lim_{n \rightarrow \infty} E\left(e^{-\frac{1}{2}t^2 B_n}\right),$$

where  $B_n$  is given by

$$\begin{aligned} B_n &= \sum_{k,l=0}^{n-1} \sum_{k',l'=0}^{n-1} f^{H_1}\left(\frac{k+1}{n}T, \frac{k'+1}{n}T\right) f^{H_2}\left(\frac{l+1}{n}T, \frac{l'+1}{n}T\right) \\ &\quad \times S^{\alpha_1, \alpha_2}(\Delta_{k,l}) S^{\alpha_1, \alpha_2}(\Delta_{k',l'}). \end{aligned}$$

Let now  $\mu_{k,j}$ ,  $k \geq 1$ , the eigenvalues of the operator  $K_{f^{H_r}}^{\alpha_r}$ ,  $r = 1, 2$ , and  $g_{k,r}$ ,  $k \geq 1$ ,  $r = 1, 2$ , the corresponding eigenfunctions. Then,  $\{g_{k,r}, k \geq 1\} \subset |\mathcal{H}|_{\alpha_r}$  and the sequence  $\{\mu_{k,r}, k \geq 1\}$  is square-summable for  $r = 1, 2$ , and moreover,  $g_{j,1} \otimes g_{k,2} \in |\mathcal{H}|(\alpha_1, \alpha_2)$ ,  $j, k \geq 1$  are orthonormal, which implies that random variables

$$B_{j,k} := \int_0^T \int_0^T g_{j,1}(s)g_{k,2}(t)dS^{\alpha_1, \alpha_2}(s, t), \quad j, k \geq 1,$$

are independent standard normal random variables. By (1) of Lemma 3.2, it follows that

$$f^{H_j}(x, y) = \sum_{k \geq 1} \mu_{k,j} g_{k,j}(x)g_{k,j}(y),$$

for  $j = 1, 2$ , and

$$B_n = \sum_{j,k \geq 1} \mu_{j,1} \mu_{k,2} \left( \sum_{l,l'=0}^{n-1} g_{j,1}\left(\frac{l+1}{n}T\right)g_{k,2}\left(\frac{l'+1}{n}T\right)S^{\alpha_1, \alpha_2}(\Delta_{l,l'}) \right)^2.$$

On the other hand, it is not difficult to check that

$$\begin{aligned} \sum_{l,l'=0}^{n-1} g_{j,1}\left(\frac{l+1}{n}T\right)g_{k,2}\left(\frac{l'+1}{n}T\right)S^{\alpha_1, \alpha_2}(\Delta_{l,l'}) &\rightarrow \\ &\int_0^T \int_0^T g_{j,1}(s)g_{k,2}(t)dS^{\alpha_1, \alpha_2}(s, t), \end{aligned}$$

in  $L^1$ , for all  $j, k \geq 1$ , as  $n$  tends to infinity. It follows that

$$B_n \rightarrow \sum_{j,k \geq 1} \mu_{j,1} \mu_{k,2} (B_{j,k})^2,$$

in  $L^1$ , as  $n$  tends to infinity, and the theorem follows. □

**Theorem 4.2.** *Assume that  $H_r > \frac{1}{2}$  and  $\alpha_r \in (0, \frac{1}{2})$ ,  $r = 1, 2$ . Then*

$$E(e^{itU}) = \prod_{j,k \geq 1} \sqrt{\frac{1}{1 + t^2 \lambda_{j,1} \lambda_{k,2}}}, \tag{4.6}$$

where the random variable  $U$  is given by (4.3) and  $\lambda_{k,r}$ ,  $k \geq 1$  are the eigenvalues of the operator  $K_{f^{\alpha_r}}^{H_r}$  with  $f^{\alpha_r}$  given by (3.5), for  $r = 1, 2$ .

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